ISRAEL JOURNAL OF MATHEMATICS **160** (2007), 157–171
DOI: 10.1007/s11856-007-0059-x

GENERALIZED CONVEX KERNELS OF SIMPLY CONNECTED L_n SETS IN THE PLANE

BY

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ABSTRACT

Let D be a compact, simply connected subset of $\Re^2$. Assume that every two points of D can be connected by a polygonal line with at most n edges within D. Then there is a point $q \in D$ that can be connected to any other point in D by a polygonal line with at most $\lceil \frac{n+1}{2} \rceil$ edges. This is best possible for all n.

1. Introduction

Let D be a subset of $\Re^2$. For points $a, b \in D$ we denote by $\rho_D(a, b)$ the smallest number of edges of a polygonal path in D that connects a and b. ($\rho_D(a, b) = \infty$ if there is no such path). When no confusion will arise, we will simply write $\rho(a, b)$ instead of $\rho_D(a, b)$. When D is polygonally connected, $\rho(\cdot, \cdot)$ is an integer valued metric on D.

1. $\rho(a, a) = 0$
2. $\rho(a, b) > 0$ for $a \neq b$
3. $\rho(a, b) = \rho(b, a)$
4. $\rho(a, c) \leq \rho(a, b) + \rho(b, c)$

The **polygonal diameter** of D ($\rho - \mathrm{diam}(D)$) is defined by

$$\sup\{\rho(a, b) : a, b \in D\}.$$

D is called an $\mathbf{L_n}$**-set** if $\rho - \mathrm{diam}(D) \leq n$. Similarly, the ρ-radius of D is defined by $\rho - \mathrm{rad}(D) = \inf_{a \in D} \sup_{b \in D} \rho(a, b)$.

Received December 16, 2003 and in revised form February 27, 2006

By definition, $\rho-\mathrm{rad}(D) \le \rho-\mathrm{diam}(D) \le 2\cdot\rho-\mathrm{rad}(D)$. We call a point a in D **central** if $\rho(a,b) \le \rho-\mathrm{rad}(D)$ for all $b \in D$.

It is well-known that if T is a finite tree, with the natural associated metric on its vertices, then $\mathrm{rad}(T) = \lceil 1/2\,\mathrm{diam}(T)\rceil$, and T has a unique central vertex when $\mathrm{diam}(T)$ is even, and two adjacent central vertices when $\mathrm{diam}(T)$ is odd.

One can easily verify that if D is the boundary of a convex $(2m-1)$-gon $(m \ge 2)$, then $\rho-\mathrm{diam}(D) = \rho-\mathrm{rad}(D) = m$, and every point of D is central.

Our aim is to show that for simply connected sets in $\Re^2$, the ρ-radius cannot be much larger than one half of the ρ-diameter. To be specific, we shall prove

THEOREM 1.1: *If D is a compact, simply connected L_n-set in $\Re^2$, then there is a point $q \in D$ such that $\rho(q,x) \le \lceil \frac{n+1}{2} \rceil$ for all $x \in D$.*

Define the ***m*-kernel** (or m-mirador) of D as follows:

$$\mathrm{Mir}_m\, D = \{x \in D : (\forall y \in D)\rho(x,y) \le m\}.$$

Using this notion we can reformulate Theorem 1.1 as follows: If $D \subset \Re^2$ is a compact, simply connected L_n-set, the $\mathrm{Mir}_{\lceil \frac{n+1}{2} \rceil} D \ne \emptyset$.

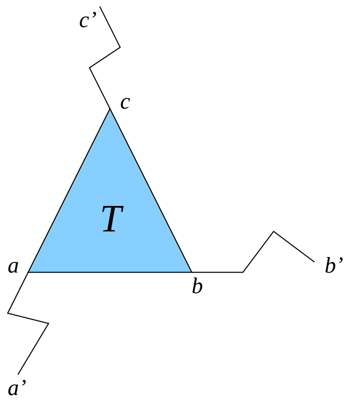
The proof of Theorem 1.1 is based on Helly's Topological Theorem (in the plane). This theorem states that a family F of compact and simply connected subsets of the plane has nonempty intersection, provided that every three members of F meet, and every two have a connected intersection. (See [H].)

We shall actually need a restricted version of this theorem (Theorem 1.2 below), which uses the notion of **relative convexity**. (A subset K of D is **relatively convex** with respect to D if for every two points $x, y \in K$, $[x,y] \subset D$ implies $[x,y] \subset K$.)

THEOREM 1.2: *Let D be a simply connected set in $\Re^2$, and let $\{K_i : i \in I\}$ with $|I| \ge 3$ be a family of subsets of D that are compact, polygonally connected and relatively convex in D, and such that each three have a point in common. Then the intersection of the whole family is nonempty.*

(For an elementary proof of this theorem, see Theorem 3.1 of [MP1].)

To show that our result is best possible consider the following example (See figure):



D_k $(k \geq 1)$ consists of a triangle $T = conv\{a, b, c\}$ and three "tails" T_a, T_b, T_c. Each tail is a simple polygonal path consisting of k edges. The first edge of T_a is an extension of the edge $[c, a]$ of T beyond a, no two consecutive edges of T_a are collinear, $T_a \cap T = \{a\}$. Similarly for T_b and T_c. The tails T_a, T_b, T_c are pairwise disjoint. D_k is an L_{2k}-set, but the ρ-radius of D_k is exactly $k+1$, not k. (The subset of D_k that can be reached from a' in at most k steps is precisely $T_a \cup [a, c]$. Similarly $T_b \cup [b, a]$ for b' and $T_c \cup [c, b]$ for c'. But the intersection of these three sets is empty.) If we replace D_k by $D_k + \varepsilon B(0, 1)$ for some sufficiently small positive ε, we get a set which is the closure of its interior and has the same ρ-diameter and ρ-radius as D_k.

The proof of Theorem 1.1 consists of three steps: In the first step we consider three points a, b, c of D with $\rho(a, b) \leq 2m-1$, $\rho(b, c) \leq 2m-1$, $\rho(c, a) \leq 2m-1$, and find a point $q \in D$ such that $\rho(a, q) \leq m$, $\rho(b, q) \leq m$, $\rho(c, q) \leq m$. In the second step we consider the sets

$$K_m(x) = \{y \in D : \rho(x, y) \leq m\} \quad (x \in D).$$

These sets have been investigated by Bruckner and Bruckner in [BB], and by us in [MP1]. They are compact, polygonally connected (actually, they are L_{2m}-sets) and relatively convex in D. (The compactness of $K_m(x)$ is a simple consequence of the compactness of D. For the relative convexity of $K_m(x)$, see Theorem 2.4 and Corollary 2.6 in [MP1], or Lemma 1 and Lemma 2 in [BB].) By step 1 every three of these sets meet.

The last step consists of applying Helly's Topological Theorem (actually, the restricted version, Theorem 1.2 above) to the sets $K_m(x)$, showing that $\bigcap_{x \in D} K_m(x) \neq \emptyset$. Now any point $q \in \bigcap_{x \in D} K_m(x)$ satisfies $\rho(q, x) \leq m$ for all $x \in D$, which concludes the proof.

The following proposition and Section 2, consist of a detailed exposition of step 1.

We start with a simple proposition:

PROPOSITION 1.3: *Suppose that $a_1, a_2, \ldots, a_t$ and q are points in D, and $\rho(a_i, q) + \rho(a_j, q) \leq 2m$ for all $1 \leq i < j \leq t$. Then there is a point $p \in D$ such that $\rho(a_i, p) \leq m$ for $i = 1, \ldots, t$.*

Proof: Rearrange the points a_1,a_2,...,a_t so that $(\rho(a_i, q))_{i=1,\ldots,t}$ is a non-increasing sequence. If $\rho(a_1, q) \leq m$ take $p = q$. If $\rho(a_1, q) > m$, assume that $\rho(a_1, q) = m + \varepsilon$. Then $\rho(a_i, q) \leq m - \varepsilon$ for $i = 2, \ldots, t$. Draw a polygonal path in D with $m + \varepsilon$ edges from q to a_1. Denote by p the ε-th vertex of that path, starting from q. Then $\rho(a_1, p) = m$, and for $2 \leq i \leq t$ we have $\rho(p, a_i) \leq \rho(p, q) + \rho(q, a_i) \leq \varepsilon + (m - \varepsilon) = m$.

2. The Main Lemma

LEMMA 2.1: *Suppose $D \subset \Re^2$ is compact and simply connected. Assume a, b and c are points in D, with $\rho(a, b) \leq n$, $\rho(b, c) \leq n$ and $\rho(c, a) \leq n$ $(n \geq 1)$. Then there is a point $q \in D$ with $\rho(a, q) \leq m$, $\rho(b, q) \leq m$ and $\rho(c, q) \leq m$, where $m = \lceil \frac{n+1}{2} \rceil$.*

This lemma is close in spirit to the following well-known fact about trees (see [Ha]):

PROPOSITION 2.2: *Let T be a tree of diameter n. Then there exists a vertex q in T such that the distance from q to any other vertex of T is $\leq \lceil \frac{n}{2} \rceil$.*

Before presenting a proof of the lemma, we state here a well-known result that will be useful for our proof (see [Hi]).

PROPOSITION 2.3: *Assume that $\mathbf{P} = \langle p_0, p_1, \ldots, p_m = p_0 \rangle$ is a simple closed m-gon in $\Re^2$. Then $\mathbf{P} \cup \operatorname{int} \mathbf{P} (= \mathrm{P})$ can be triangulated by pairwise noncrossing interior diagonals. Each such triangulation has $m - 2$ triangles and uses $m - 3$ diagonals.*

Proof of Lemma 2.1:

Remark: If $m = \lceil \frac{n+1}{2} \rceil$, then $n = 2m - 2$ or $n = 2m - 1$. If $n = 1$ or 2, then $m = n$. In this case, any of the three given points a, b or c can serve as q. We assume, therefore, that $n \geq 3$.

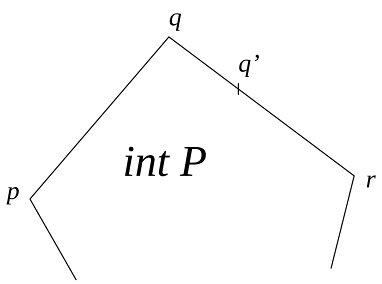
We also assume that

$$\rho(a,b) \le 2m-1, \quad \rho(b,c) \le 2m-1 \quad \text{and} \quad \rho(c,a) \le 2m-1.$$

The **total length** of a polygonal path is defined as the sum of the (euclidean) lengths of its edges. Among all paths in D with $\rho(a,b)$ edges that connect a and b, there exists one of minimal total length. This follows easily from the compactness of D. This will be called a **shortest** path from a to b. There may be more than one shortest path (see section 3).

Let x, y be two points in D. We choose a shortest path in D from x to y and denote it by $\mathrm{L}(x,y)$. Put $\mathrm{L}_1 = \mathrm{L}(a,b)$, $\mathrm{L}_2 = \mathrm{L}(b,c)$ and $\mathrm{L}_3 = \mathrm{L}(c,a)$. Each couple of these paths share an endpoint, but they may intersect in other points as well. The idea is to find a polygon P (usually not convex) that is bounded by some sections of the three paths $\mathrm{L}_1, \mathrm{L}_2$ and L_3, and find the required point q inside, or on the boundary of P.

Let us first consider the **simple scenario**: The three paths meet only at their common endpoints. In this case the union of these three paths is a simple closed polygon $\mathbf{P} \subset D$. The interior of $\mathbf{P}$ (i.e., the bounded component of $\Re^2 - \mathbf{P}$) is in D, since D is simply connected. Henceforth we denote by P the union of $\mathbf{P}$ and its interior. First we observe that the interior angle of $\mathbf{P}$ at any vertex other than a, b or c is $> 180°$. In fact, suppose q is an interior vertex of one of the three paths, say of L_1. Denote by p and r the two vertices of L_1 that are adjacent to q (see the figure below).



Since L_1 has exactly $\rho(a,b)$ edges, the points p, q and r cannot be collinear. If the interior angle of $\mathbf{P}$ at q is $< 180°$, then we mark a point q' inside the edge $[q,r]$ in L_1. If q' is sufficiently close to q, then the segment $[p,q']$ lies entirely in P, hence in D. Thus we can replace the edges $[p,q]$ and $[q,r]$ of L_1 by $[p,q']$ and $[q',r]$, thereby creating a metrically shorter path, with the same number of edges, contrary to our choice of L_1. (Note that if $\mathbf{P}$ is a simple closed polygon with n edges, then the sum of the interior angles at the vertices of $\mathbf{P}$ is exactly

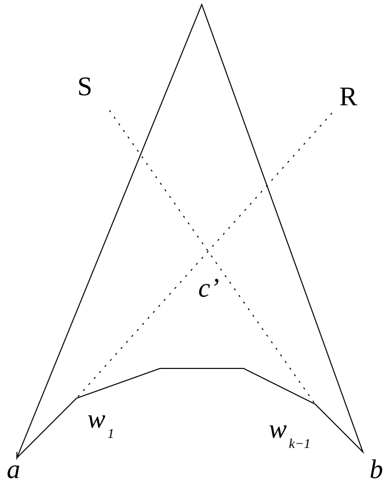
$(n-2)\cdot 180°$. It follows that at most $n-3$ of the interior angles can be $\geq 180°$, i.e, at least three are less than $180°$.)

Next we show that each of the three paths that form **P** has at most two edges. But this means that we are in the case $n \leq 2$, which is already settled. Assume on the contrary that L_1 has more than two edges. Let $\mathrm{L}_1 = \langle a, w_1, w_2, \ldots, w_{k-1}, b\rangle$ with $k \geq 3$. Extending the segment $[a, w_1]$ beyond w_1 we obtain the ray R, as shown in the figure below. This ray starts inside P, because the interior angle of **P** at w_1 is greater than $180°$. Eventually this ray has to leave P through a point of **P**. Denote by a' the first point of R (excluding w_1) that belongs to **P**. We claim that a' is an interior point of the path L_2, or, in other words, $a' \notin \mathrm{L}_1$ and $a' \notin \mathrm{L}_3$.

Indeed, if $a' \in \mathrm{L}_1$ then the union of the part of L_1 between w_1 and a' and the segment $[a', w_1]$ is a simple closed polygon $\mathbf{Q} \subset \mathrm{P}$. The interior angle of **Q** at each vertex, except w_1 and a', is greater than $180°$, which is impossible. If $a' \in L_3$, then we apply the same argument to the polygon $\mathbf{Q}'$ that consists of the segment $[a, a']$ and the part of L_3 between a' and a.

By the same token, if we extend the segment $[b, w_{k-1}]$ beyond w_{k-1} we get a ray S that starts inside P and meets **P** for the first time in a point b' which is interior to L_3.

The two open segments (w_1, a') and (w_{k-1}, b') lie entirely in the interior of P $(\subset D)$, and the two pairs of endpoints separate each other on **P**. Therefore the two segments must cross at a point $c' \in \operatorname{int} \mathrm{P}$, as shown in the figure below. But this means that a and b can be connected by the path $[a, c'] \cup [c', b]$ which lies entirely in P, contradicting our assumption that $\rho(a, b) > 2$.



Now we pass to the **general scenario**: At least two of the paths L_1, L_2 and L_3 intersect in points other than their common endpoint.

If $a \in \mathrm{L}_2$ we are done (q can be chosen as a central vertex of L_2). So we may assume that none of the points a, b or c belongs to the path joining the other two.

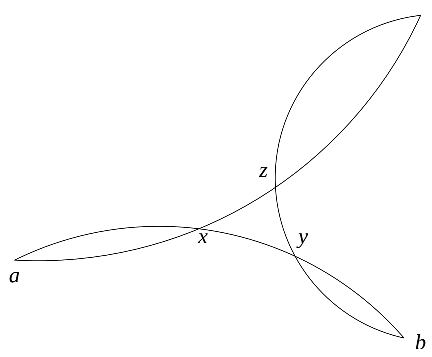
Denote by x the first point on L_3 that belongs to L_1, by y the first point on L_1 that belongs to L_2 and by z the first point on L_2 that belongs to L_3. (Here we regard L_3 as directed from c to a, L_1 from a to b and L_2 from b to c, as shown in the figure below.) In the simple scenario, $x = a, y = b$ and $z = c$. We now proceed to establish some inequalities involving the ρ-distances between the points a, b and c on one hand, and the newly defined points x, y and z on the other hand. Besides the ρ-distances we will also use the distances along the paths L_1, etc. If $t \in \mathrm{L}_1$, then $\rho_1(a,t)$ denotes the number of edges of the part of L_1 between a and t. Similarly for $\rho_1(t,b)$.

Clearly $\rho_1(a,t) \geq \rho(a,t)$, and the minimality of the path L_1 implies $\rho_1(a,t) \leq \rho(a,t)+1$. Moreover, $\rho_1(a,t)+\rho_1(t,b)$ equals either $\rho(a,b)$ (when t is a vertex of L_1), or $\rho(a,b)+1$ (when t is an interior point of an edge in L_1).

We obtain the following inequalities:

1. $\rho(a,x)+\rho(x,b) \leq \rho_1(a,x)+\rho_1(x,b) \leq 2m$, and the last inequality is always strict if x is a vertex of L_1.
2. $\rho(a,y)+\rho(y,b) \leq \rho_1(a,y)+\rho_1(y,b) \leq 2m$, and the last inequality is always strict if y is a vertex of L_1.
3. $\rho(b,y)+\rho(y,c) \leq \rho_2(b,y)+\rho_2(y,c) \leq 2m$, and the last inequality is always strict if y is a vertex of L_2.
4. $\rho(b,z)+\rho(z,c) \leq \rho_2(b,z)+\rho_2(z,c) \leq 2m$, and the last inequality is always strict if z is a vertex of L_2.
5. $\rho(c,z)+\rho(z,a) \leq \rho_3(c,z)+\rho_3(z,a) \leq 2m$, and the last inequality is always strict if z is a vertex of L_3.
6. $\rho(c,x)+\rho(x,a) \leq \rho_3(c,x)+\rho_3(x,a) \leq 2m$, and the last inequality is always strict if x is a vertex of L_3.

We say that the points x, y and z appear "in the right order" on the respective paths if x precedes y on L_1, y precedes z on L_2 and z precedes x on L_3. That is, our paths look like the schematic drawing in the figure below.



If this happens, then of course

$$\rho_1(a,x) \le \rho_1(a,y), \quad \rho_2(b,y) \le \rho_2(b,z), \quad \rho_3(c,z) \le \rho_3(c,x).$$

We are going to show that the case where x, y and z appear in the right order is the only one that deserves further attention. In all other cases, our lemma can be easily deduced. In fact, if some pair is not in the right order, e.g., if z precedes y on L_2 (or if $z = y$), then, by the Monotonicity Lemma ([MP1], Lemma 2.8), $\rho(b,z) \le \rho(b,y)$ and $\rho(y,c) \le \rho(z,c)$. This enables us to use Proposition 1.3, as follows:

In case $\rho(a,z) \le \rho(a,y)$ we have

$$\rho(a,z) + \rho(b,z) \le \rho(a,y) + \rho(b,y) \le 2m \quad \text{by ineq. 2},$$
$$\rho(a,z) + \rho(z,c) \le 2m \quad \text{always},$$
$$\rho(b,z) + \rho(z,c) \le 2m \quad \text{always}.$$

By Proposition 1.3 there is a point $q \in D$ such that $\rho(a,q) \le m$, $\rho(b,q) \le m$, and $\rho(c,q) \le m$, and so our lemma holds.

The case $\rho(a,z) > \rho(a,y)$ is similar, with y instead of z as the intermediate point.

Assuming then that the points appear in the right order, our paths include a simple closed polygonal line **P** consisting of three arcs. The first arc is the part of L_1 between x and y (call it $L(x,y)$), the second arc is the part of L_2 between y and z ($= L(y,z)$), and the third arc consists of the part of L_3 between z and x ($= L(z,x)$). From the definitions of x, y and z it follows that these three arcs meet only at their common endpoints. We know that $\rho(b,y) + \rho(y,c) \le 2m$ and $\rho(b,y) + \rho(y,a) \le 2m$. If also $\rho(a,y) + \rho(y,c) \le 2m$, then our lemma holds by Proposition 1.3. So assume on the contrary that

(**A**) $$\rho(a,y) + \rho(y,c) \ge 2m + 1.$$

Repeating the same argument with the points a, b, c and x, y, z cyclically permuted, we may also assume that

$$\text{(B)} \qquad \rho(b,x) + \rho(x,c) \geq 2m+1$$

and

$$\text{(C)} \qquad \rho(a,z) + \rho(z,b) \geq 2m+1.$$

Each of the three arcs in **P** may be either concave (i.e., at each interior vertex of that arc, the interior angle of **P** is greater than 180°) or not. An arc with only one edge is also considered concave.

We know that if one of the arcs $\mathrm{L}(x,y)$, $\mathrm{L}(y,z)$ or $\mathrm{L}(z,x)$ is not concave, then it consists of two incomplete edges of the corresponding complete arc (L_1, L_2 or L_3). (This is an argument that we have already used in the simple scenario. If, for example, q is an interior vertex of $\mathrm{L}(x,y)$, $\mathrm{L}(x,y)$ is convex at q, and the neighbours of q in L_1 are p and r, and at least one of them (say p) is also in $\mathrm{L}(x,y)$, then we can replace q by a point $q' = (1-\epsilon)q + \epsilon r$ ($\epsilon > 0$ small), thus producing another $\rho(a,b)$-path $\langle a, \ldots, p, q', r, \ldots, b\rangle$ in D from a to b that is metrically shorter than L_1. But L_1 was chosen to be metrically shortest!)

We shall distinguish the following four cases:

I. 3 concave arcs.
II. 2 concave arcs, and one not.
III. 1 concave arc, and two not.
IV. no concave arc.

Adding inequalities 2 and 3 we obtain

$$\begin{gathered}\rho_1(a,y) + \rho_1(y,b) + \rho_2(b,y) + \rho_2(y,c) \leq 4m \\ \Rightarrow \rho_1(b,y) + \rho_2(b,y) \leq 4m - \rho_2(y,c) - \rho_1(a,y).\end{gathered}$$

By **(A)** we have:

$$(*) \qquad \rho_1(b,y) + \rho_2(b,y) \leq 4m - (2m+1) \leq 2m-1.$$

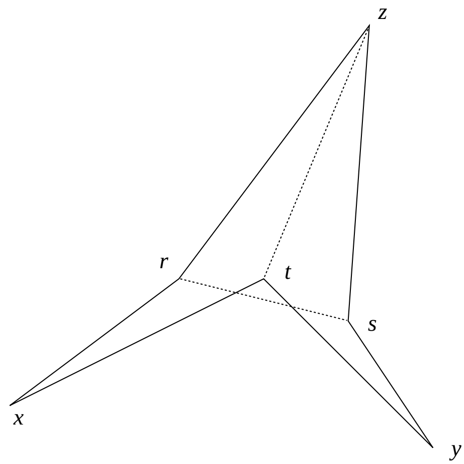
This inequality certainly implies $2\rho(b,y) \leq 2m-1$, or $\rho(b,y) \leq m-1$.

The same inequalities can be derived both for $\rho_3(c,z) + \rho_2(c,z)$ and for $\rho_3(a,x) + \rho_1(a,x)$, thus leading to

$$(**) \qquad \rho(x,a) \leq m-1, \quad \rho(y,b) \leq m-1 \quad \text{and} \quad \rho(z,c) \leq m-1.$$

CASE I: If all three arcs $\mathrm{L}(x,y)$, $\mathrm{L}(y,z)$ and $\mathrm{L}(z,x)$ are concave, then each of them has at most two edges, so **P** has at most six edges. This follows again from the arguments presented in the simple scenario.

If the polygon P happens to be starshaped and q is any point in the kernel of P, then q sees x, y and z only in one step. By (**), q sees a, b and c in at most m steps, and we are done. Since every polygon with fewer than 6 sides is starshaped, we may as well assume that **P** is a hexagon.



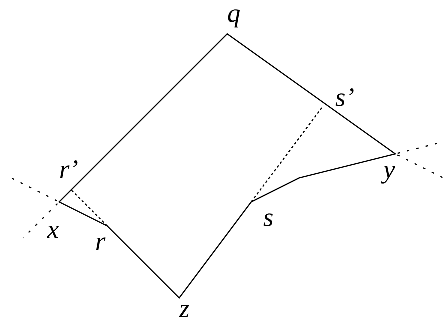
If $t \in [r,s]$ or $[r,s]$ is not completely in P, as in the figure above, then $[t,z] \subset \mathrm{P}$, P is starshaped with respect to t, and we are done. We may therefore assume that $[r,s]$, $[s,t]$ and $[t,r]$ are interior diagonals of P, and thus $\mathrm{conv}\{r,s,t\} \subset \mathrm{P}$ ($\subset D$). Any point $q \in \mathrm{conv}\{r,s,t\}$ sees r, s and t in one step (via D). Thus it suffices to show that each of the points a, b, c sees r, s or t in at most $m-1$ steps.

Let us do this for b. We have to distinguish three possibilities concerning the point $y \in \mathrm{L}_1 \cap \mathrm{L}_2$.

1. y is an interior point of an edge in L_2 and also an interior point of an edge in L_1.
   In this case, since $\rho_1(b,y) + \rho_2(b,y) \leq 2m-1$, one of these numbers is $\leq m-1$. If $\rho_1(b,y) \leq m-1$, then $\rho_1(b,y) = \rho_1(b,t) \leq m-1$, and if $\rho_2(b,y) \leq m-1$, then $\rho_2(b,y) = \rho_2(b,s) \leq m-1$.
2. y is an interior point of an edge in L_2 and a vertex in L_1 (or vice versa). In this case the inequality (*) is strict. Thus either $\rho_2(y,b) \leq m-1$, in which case we continue as in 1, or $\rho_1(b,y) \leq m-2$, which certainly implies $\rho(b,s) \leq m-1$.

3. y is a vertex in both paths. In this case $\rho_1(b, y) + \rho_2(b, y) \leq 2m - 3$, since both inequalities 2 and 3 are strict, and thus one of these numbers must be $\leq m - 2$, which again implies $\rho(b, s) \leq m - 1$.

Case II:



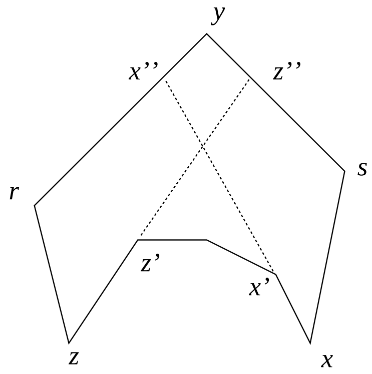
Assume $L(x, y) = \langle x, q, y \rangle$ is not concave. The point q satisfies $\rho(a, q) \leq m$ and $\rho(b, q) \leq m$, since $\rho(a, x) \leq m - 1$ and $\rho(b, y) \leq m - 1$. If we continue the segment $[z, r]$ beyond r, by our minimality assumptions, it must leave P at a point of $[x, q]$ or at a point of $[q, y]$. Call this point r'. In a similar way we define s' (see the figure above). (It may happen that $s = s' = y$, or $r = r' = x$, or both).

Again, we distinguish three possibilities:

1. z is an interior point of an edge in L_2, and also an interior point of an edge in L_3. In this case, the inequality $\rho_2(z, c) + \rho_3(z, c) \leq 2m - 1$ implies either $\rho_2(s, c) = \rho_2(z, c) \leq m - 1$ or $\rho_3(r, c) = \rho_3(z, c) \leq m - 1$. Assume, e.g., that the first one holds. Since $[s, z] \subset [s', z] \subset P \subset D$, we conclude that $\rho(c, s') \leq m - 1$ as well. But q sees s', hence $\rho(c, q) \leq m$.
2. z is an interior point of an edge in L_2, and a vertex in L_3 (or vice versa). In this case $\rho_2(c, z) + \rho_3(c, z) \leq 2m - 2$. If $\rho_2(c, s) = \rho_2(c, z) \leq m - 1$, then we proceed as in 1. If not, then $\rho_3(c, z) \leq m - 2 \Rightarrow \rho(c, s') \leq m - 1 \Rightarrow \rho(c, q) \leq m$.
3. z is a vertex in both paths. Then $\rho(c, z) \leq m - 2 \Rightarrow \rho(c, s') \leq m - 1 \Rightarrow \rho(c, q) \leq m$.

Note that, in spite of the figure, in Case II z does not necessarily see q via P.

CASE III:



Assume that $L(x, y) = \langle x, s, y\rangle$ and $L(y, z) = \langle y, r, z\rangle$ are not concave. If the concave arc $L(x, z)$ has only one edge, then P is a starshaped pentagon and we are done. Therefore, we may assume that $L(x, z)$ has at least two edges.

Let us first assume that it has more than two edges (see the figure above).

In this case, z is an interior point of an edge in L_2 and x is an interior point of an edge in L_1. If we continue the segment $[z, z']$ beyond z', it must hit $\mathbf{P}$ again (for the first time) at a point z'' that belongs to one of the two convex arcs. In the same way we define x''. If $[z, z'']$ and $[x, x'']$ intersect within P, then we have a contradiction to our minimality assumptions. This happens when $x'' \leq z''$ in the directed path $\langle z, r, y, s, x\rangle$. Thus we must have $x'' > z''$, which implies that both z'' and x'' are on $[r, y] \cup [y, s]$, and so both see y. Let us try to prove our lemma in this case by showing that $\rho(a, y) \leq m$, $\rho(b, y) \leq m$ and $\rho(c, y) \leq m$. We will only prove in detail that $\rho(a, y) \leq m$. $\rho(c, y) \leq m$ is done in the same manner (our assumptions are symmetric in a and c). As for b, $\rho(b, y) \leq m - 1$ by (**).

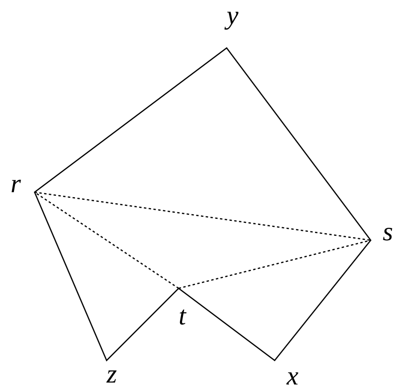
Recall the inequality $\rho_1(a, x) + \rho_3(a, x) \leq 2m - 1$, which is strict if x is a vertex of L_3. This leads to (at least) one of the following three conclusions:

α. $\rho_1(a, x) \leq m - 1$, or

β. $\rho_3(a, x) \leq m - 1$ and x is an interior point of an edge of L_3, or

γ. $\rho_3(a, x) \leq m - 2$ and x is an vertex of L_3.

If (α) holds then $\rho(a, y) \leq 1+\rho(a, s) \leq 1+\rho_1(a, s) = 1+\rho_1(a, x) \leq m$. (Note that x is an interior point of an edge of L_1.) If (β) holds then $\rho(a, y) \leq 1+\rho(a, x'') \leq 1 + \rho_3(a, x') = 1 + \rho_3(a, x) \leq m$. If ($\gamma$) holds then clearly $\rho(a, y) \leq m$.



If the concave arc $L(x, z)$ has only two sides, then $\mathbf{P}$ is a hexagon (see the figure above), convex (i.e., interior angle is less than 180°) at r and s, and concave at t. If $[y, t] \subset P$, $[r, x] \subset P$ or $[s, z] \subset P$, then our lemma holds with $q = t$, $q = r$ or $q = s$ respectively, in view of (**). Therefore, we may assume that $[t, y], [r, x]$ and $[s, z]$ are not interior diagonals of P. Note also that $[x, z]$ is not an interior diagonal of P. By Proposition 2.3, P has a triangulation by three pairwise non-crossing interior diagonals. The only remaining triple of pairwise non-crossing interior diagonals of P is $[r, s]$, $[s, t]$ and $[t, r]$, as shown in the figure above. Since both $\langle y, r, z\rangle$ and $\langle x, s, y\rangle$ are not concave, y is an interior point of an edge in L_1 and also in L_2, and therefore $\rho_1(b, y) = \rho_1(b, s) \geq \rho(b, s)$ and $\rho_2(b, y) = \rho_2(b, r) \geq \rho(b, r)$. By (*), $\rho_1(b, y) + \rho_2(b, y) \leq 2m - 1$, which implies $\min(\rho(b, s), \rho(b, r)) \leq m - 1$. Thus our lemma holds with $q = t$.

Case IV: There are no concave arcs.

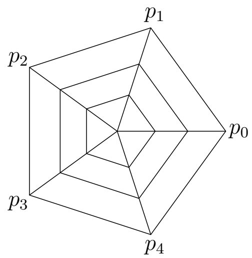
In this case $\mathbf{P} = \langle x, s, y, r, z, t, x\rangle$ is a hexagon, locally convex (at least) at the vertices r, s, t. If P has an interior main diagonal, say $[r, x]$, then our lemma holds with $q = r$. If not, then there are two possible triangulations of P by pairwise noncrossing diagonals: Either $[x, y], [y, z]$ and $[z, x]$, in which case our lemma holds with any point $q \in \operatorname{conv}\{x, y, z\}$, or $[r, s], [s, t]$ and $[t, r]$. In this last case P is locally convex at the vertices x, y and z as well, and therefore convex (by Tietze's theorem).

As these cases are exhaustive, we have proved our statement.

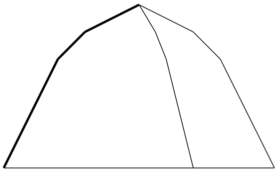
3. Concluding remarks

The higher dimensional case. Theorem 1.1 fails completely in dimension 3. In fact, for each $m \geq 2$ we can produce a simply connected set

$D = D_m \subset \Re^3$ that is homeomorphic to a two dimensional disc and satisfies $\rho - \text{diam}(D) = \rho - \text{rad}(D) = m$. A description follows:



Side View



Top View

For $i = 0, 1, \ldots, 2m-1$ define points $p_i = (\cos\frac{2\pi i}{2m-1}, \sin\frac{2\pi i}{2m-1}, 0)$ ($p_{2m-1} = p_0 = (1,0,0)$). The points $p_1, \ldots, p_{2m-1}$ are the vertices of a regular $(2m-1)$-gon inscribed in the equator of the unit sphere in $\Re^3$. Put $q_0 = (0,0,1)$ (the north pole), and define for $j = 1, \ldots, m$ points $q_{i,j} = \cos(\frac{\pi j}{2m})q_0 + \sin(\frac{\pi j}{2m})p_i$ ($q_{i,m} = p_i$).

Q $= \text{conv}\{\{q_0\} \cup \{q_{i,j} : i = 1, \ldots, 2m-1; j = 1, \ldots, m\}\}$ is a 3-polytope inscribed in the northern half of the unit sphere.

The facets of Q are:

1. $2m-1$ triangles $\text{conv}\{q_0, q_{i-1,1}, q_{i,1}\}$ ($i = 1, 2, \ldots, 2m-1$).
2. $(2m-1)(m-1)$ trapezoids

$$\text{conv}\{q_{i-1,j}, q_{i,j}, q_{i-1,j+1}, q_{i,j+1}\}$$

(for $i = 1, 2, \ldots, 2m-1; j = 1, \ldots, m-1$).

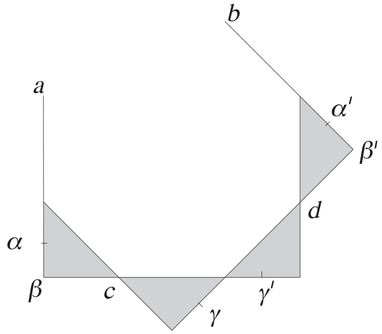
3. The base, the regular $(2m-1)$-gon $\text{conv}\{p_i : i = 1, 2, \ldots, 2m-1\}$.

D is the union of all facets of Q, except the base.

$D = D_m$ is an L_m-set, and for each point $x \in D$ there is another point $y \in D$ such that $\rho(x,y) = m$. (Hint: if $x = q_0$, take y to be any point on the boundary of the base. If $x \neq q_0$, take y to be a point on the boundary of the base "opposite" x.) We leave the detailed verification to the reader.

THE NON-COMPACT CASE. Our proof of Theorem 1.1 depends heavily on the compactness of D. It may be, though, that Theorem 1.1 holds for any simply connected set $D \subset \Re^2$, or perhaps at least for closed (but unbounded) sets D. Our proof can be modified to show at least the following: *If $D \subset \Re^2$ is simply connected, if $F \subset D$ is finite, and $\rho(x,y) \leq 2m-1$ holds for all $x, y \in F$, then there is a point $q \in D$ such that $\rho(x,q) \leq m$ for all $x \in F$*

Non-uniqueness of the minimal path. Consider the set shown in the following figure.



D is the union of four congruent full triangles with side lengths $1, 1, \sqrt{2}$, and two unit segments. D is an L_4-set. One type of 4-path from a to b goes from a to α, then crosses through c to γ, continues to β' and from there to b. Using some calculus, one can show that such a path is metrically shortest when $[\alpha, \gamma]$ bisects the angles at c. The mirror image of this path ($= \langle b, \alpha', \gamma', \beta, a \rangle$) is another shortest 4-path from a to b.

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